POINCARÉ-LELONG EQUATION VIA THE HODGE LAPLACE HEAT EQUATION

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1. INTRODUCTION

Solving the Poincaré-Lelong equation amounts to, for a given real $(1,1)$-form ρ, finding a smooth function u such that $\sqrt{-1}\partial\bar{\partial}u = \rho$. Motivated by geometric considerations, on a complete noncompact Kähler manifold (M, g), this was first studied by Mok-Siu-Yau [9] under some restrictive conditions including a point-wise quadratic decay on $\|\rho\|$, nonnegative bisectional curvature and maximum volume growth on M. There have been many works since then. Finally in [13], the following result was proved.

Theorem 1.1. *Let M^n (with $m = 2n$ being the real dimension) be a complete Kähler manifold with nonnegative holomorphic bisectional curvature. Let ρ be a real d-closed $(1,1)$-form. Suppose that*

$$(1.1) \qquad \int_0^\infty \left(\fint_{B_o(s)} \|\rho\|(y)d\mu(y) \right) ds < \infty,$$

and

$$(1.2) \qquad \liminf_{r\to\infty} \left[\exp\left(-\alpha r^2\right) \cdot \int_{B_o(r)} \|\rho\|^2(y)d\mu(y) \right] < \infty$$

for some $\alpha > 0$. Then there is a solution u of the Poincaré-Lelong equation $\sqrt{-1}\partial\bar{\partial}u = \rho$. Moreover, for any $0 < \epsilon < 1$, u satisfies

$$\alpha_1 r \int_{2r}^\infty k(s)ds \;+\; \beta_1 \int_0^{2r} sk(s)ds \geq u(x)$$

$$(1.3) \qquad \geq \beta_3 \int_0^{2r} sk(s)ds - \alpha_2 r \int_{2r}^\infty k(s)ds - \beta_2 \int_0^{\epsilon r} sk(x,s)ds$$

for some positive constants $\alpha_1(m)$, $\alpha_2(m,\epsilon)$ and $\beta_i(m)$, $1 \leq i \leq 3$, where $r = r(x)$. Here $k(x,s) = \fint_{B_x(s)} \|\rho\|$ and $k(s) = k(o,s)$, where $o \in M$ is fixed.

Due to the technical nature of the assumption (1.2), which arises from the parabolic method employed in [13] and is related to the uniqueness of the heat equation solution, it is desirable to be able to remove it. The purpose of this paper is to prove, in Theorem 7.1, that the above result remains true without assuming (1.2), provided either $\rho + a\omega \geq 0$ with some constant $a \geq 0$ and ω

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being the Kähler form, or (M^n, g) has nonnegative sectional curvature outside a compact subset, or of maximum volume growth. What in fact was proved in Theorem 7.1 is a bit more. The solution space to a Poincaré-Lelong equation clearly is an affine space consisting a special solution summing the linear space of the pluriharmonic functions. The estimate (1.3) selects the minimum one in a sense. In the view that the sublinear growth is the optimal necessary condition to imply the constancy of a pluriharmonic function, the assumption (1.1) is almost the optimal condition which one can expect to ensure that estimate (1.3) selects the unique (up to a constant) solution.

The method here is motivated by that of [11], namely via the study of the Cauchy problem to the Hodge-Laplace heat equation on $(1, 1)$-forms. On the other hand the construction in Section 3 supplies the necessary argument for the proof to Proposition 3.1 of [11]. The gap theorem in [11] also follows from Theorem 7.1 of this paper and Theorem 0.2 of [13].

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2. A general method to solve the Poincaré-Lelong equation

Let (M^n, g) be a complete noncompact Kähler manifold with complex dimension n and ρ is a d-closed real $(1, 1)$-form with trace $\mathrm{tr}(\rho)$ (also denoted by $\Lambda\rho$, using the notions from the Kähler geometry). Assume M has nonnegative Ricci curvature.

Let $-\Delta = \Delta_d = (d\delta + \delta d)$ be the Hodge Laplacian for forms. The following result gives connection between solving the Poincaré-Lelong equation and the global solution to a Hodge-Laplace heat equation.

Theorem 2.1. *Suppose M has nonnegative Ricci curvature. Suppose the following are true:*

(a) *There is an $(1, 1)$-form $\eta(x, t)$ satisfying*

$$(2.1) \qquad \begin{cases} \eta_t - \Delta\eta & = 0, \ in \ M \times [0, \infty); \\ \eta(x, 0) & = \rho(x), \quad x \in M, \end{cases}$$

such that $\eta(x, t)$ is closed for all t and there is $p > 0$

$$(2.2) \qquad \lim_{R \to \infty} \frac{1}{R^2 V_o(R)} \int_0^T \int_{B_o(R)} ||\eta||^p(x, t) dx\, dt = 0$$

for all $T > 0$. Moreover, $\lim_{t \to \infty} \eta(x, t) = 0$.

(b) *There is a function $u(x)$ solving $\Delta u = \mathrm{tr}(\rho)$, where tr denotes the trace, and a solution $v(x, t)$ of*

$$(2.3) \qquad \begin{cases} v_t - \Delta v & = 0, \ in \ M \times [0, \infty); \\ v(x, 0) & = u(x), \quad x \in M, \end{cases}$$

such that

$$(2.4) \qquad \lim_{R \to \infty} \frac{1}{R^2 V_o(R)} \int_0^T \int_{B_o(R)} |v|^p(x, t) + |u(x)|^p\, dx\, dt = 0$$

for all $T > 0$ and $\lim_{t\to\infty} \partial\bar{\partial}v(x,t) = 0$.
Then $2\sqrt{-1}\partial\bar{\partial}u = \rho$.

Before we prove the theorem, let us first recall the following:

Lemma 2.1. *For any d-closed $(1,1)$-form η, we have*

$$(2.5) \qquad \partial\bar{\partial}\Lambda\eta = \sqrt{-1}\Delta_{\bar{\partial}}\eta,$$

where $\Delta_{\bar{\partial}}$ is the $\bar{\partial}$-Laplacian.

Proof. Recall that $\partial\Lambda - \Lambda\partial = -\sqrt{-1}\bar{\partial}^*$, $\bar{\partial}\Lambda - \Lambda\bar{\partial} = \sqrt{-1}\partial^*$. Since $\partial\eta = \bar{\partial}\eta = 0$ and $\partial\bar{\partial} = -\bar{\partial}\partial$ and $\Delta_\partial = \Delta_{\bar{\partial}}$, we have that

$$\begin{aligned}
\partial\bar{\partial}\Lambda\eta &= \frac{1}{2}\left(\partial\bar{\partial}\Lambda\eta - \bar{\partial}\partial\Lambda\eta\right) \\
&= \frac{1}{2}\left(\sqrt{-1}\partial\partial^*\eta + \sqrt{-1}\bar{\partial}\bar{\partial}^*\eta\right) \\
&= \frac{1}{2}\sqrt{-1}\left(\Delta_\partial\eta + \Delta_{\bar{\partial}}\eta\right) \\
&= \sqrt{-1}\Delta_{\bar{\partial}}\eta.
\end{aligned}$$

This proves the claimed identity. $\qquad\square$

We also need the following maximum principle.

Lemma 2.2. *Suppose (M^m, g) is a complete noncompact Riemannian manifold with nonnegative Ricci curvature and u is a smooth nonnegative subsolution of the heat equation on $M \times [0,T]$ such that there exists a sequence $R_i \to \infty$ and $p > 0$ such that*

$$(2.6) \qquad \lim_{i\to\infty} \frac{1}{R_i^2 V_o(R_i)} \int_0^T \int_{B_o(R_i)} u^p(y,t)dy\,dt = 0.$$

Suppose $u(x,0) = 0$ for all $x \in M$. Then $u(x,t) \equiv 0$. In particular, if u_1 and u_2 are two solutions of the heat equation such that $|u_1|$ and $|u_2|$ satisfy the decay conditions (2.6) and if $u_1 = u_2$ at $t = 0$, then $u_1 \equiv u_2$.

Proof. By [6, Theorem 1.2], there is a constant $C > 0$ independent of R such that

$$\sup_{B_o(\frac{1}{2}R)\times[0,T]} u^p \leq \frac{C}{R^2 V_o(R)} \int_0^T \int_{B_o(R)} u^p(y,t)dy\,dt$$

if $R^2 \geq 4T$. From this the first assertion follows.

To prove the second assertion, apply the above argument to $(|u_1 - u_2|^2 + \epsilon)^{\frac{1}{2}}$ for $\epsilon > 0$ and let $\epsilon \to 0$. $\qquad\square$

Proof of Theorem 2.1. Let η be as in (a). Let $\phi = \mathrm{tr}(\eta)$. Then ϕ satisfies the heat equation in $M \times [0,\infty)$ with initial value $\mathrm{tr}(\rho)$. Let

$$w(x,t) = -2\int_0^t \phi(x,s)ds.$$

Then

$$w_t - \Delta w = -2\text{tr}(\rho), \ w(x,0) = 0.$$

Hence $\tilde{v}(x,t) \doteq 2u(x) - w(x,t)$ satisfies

$$\tilde{v}_t - \Delta \tilde{v} = 0, \ \tilde{v}(x,0) = 2u(x).$$

By (2.2), (2.4) and Lemma 2.2, we conclude that $v = \tilde{V} = 2u - w$.

On the other hand, by Lemma 2.1 and the fact that η is closed, we have

$$\begin{aligned}
\frac{d}{dt}\left(\eta + \sqrt{-1}\partial\bar{\partial}w\right) &= -2\Delta_{\bar{\partial}}\eta + \sqrt{-1}\partial\bar{\partial}w_t \\
&= -2\Delta_{\bar{\partial}}\eta - 2\sqrt{-1}\partial\bar{\partial}\Lambda\eta \\
&= 0.
\end{aligned}$$

At the same time, at $t = 0$, $\eta + \sqrt{-1}\partial\bar{\partial}w(\cdot,t) = \rho$. Hence this equation holds for all t. That is to say,

$$\eta + 2\sqrt{-1}\partial\bar{\partial}u - \sqrt{-1}\partial\bar{\partial}v = \rho.$$

Since $\lim_{t\to\infty}\eta(x,t) = \lim_{t\to\infty}\sqrt{-1}\partial\bar{\partial}v(x,t) = 0$, we have $2\sqrt{-1}\partial\bar{\partial}u = \rho$. $\square$

By [13], under an average growth condition we can find u and v satisfying (b) in the theorem. First let $o \in M$ be a fixed point. For a smooth function f on M, let

(2.7)
$$k_f(r) = \frac{1}{V_o(r)}\int_{B_o(r)}|f|.$$

where $B_x(r)$ is the geodesic ball of radius r with center at x and $V_x(r)$ is the volume of $B_x(r)$. First recall the following result from [13, Lemma 1.1]:

Lemma 2.3. *Let (M, g) be a complete noncompact Riemannian manifold with nonnegative Ricci curvature and let $o \in M$ be a fixed point. Let $h \geq 0$ be a continuous function. Let*

$$v(x,t) = \int_M H(x,y,t)h(y)dy$$

where $H(x,y,t)$ is the heat kernel. Assume that v is defined on $M \times [0,T]$ for some $T > 0$, and

(2.8)
$$\liminf_{r\to\infty}\exp\left(-\frac{r^2}{20T}\right)\int_{B_o(r)}h = 0.$$

Then for $t \in (0, \min(r^2, T)]$ and $p \geq 1$

$$\frac{1}{V_o(r)}\int_{B_o(r)}v^p(x,t)dx \leq C(n,p)\left[k_{h^p}(4r) + t^{-p}\left(\int_{4r}^\infty s\exp\left(-\frac{s^2}{20t}\right)k_h(s)ds\right)^p\right].$$

Proof. Note the proof in [13] (page 467-468) can be carried over because only (2.8) is needed for the integration by parts. $\square$

Proposition 2.1. *Let (M, g) be a complete noncompact Riemannian manifold with nonnegative Ricci curvature and let $o \in M$ be a fixed point. Let f be a smooth function on M such that*

$$\int_0^\infty k_f(r)dr < \infty.$$

Then we can find functions u and v with $\Delta u = f$, v satisfying (2.3) such that (2.4) is true for $p = 1$, and $\lim_{t \to \infty} \partial \bar{\partial} u(x, t) = 0$. Moreover u satisfies (1.3). In fact, $u(x)$ and $v(x, t)$ are given by

$$
\begin{aligned}
u(x) &= \int_M \left(G(o, y) - G(x, y) \right) f(y) dy, \text{ if } M \text{ is nonparabolic} \\
v(x, t) &= \int_M H(x, y, t) u(y) dy.
\end{aligned}
$$

Here $G(x, y)$ is the minimal positive Green's function of M (if M is non-parabolic) and $H(x, y, t)$ is the heat kernel of M.

Proof. First consider the case that M is nonparabolic. The existence of u and v are given by the above expressions. The claim $\lim_{t \to \infty} \partial \bar{\partial} v(x, t) = 0$ follows from [13, Lemma 6.1]. By the proof of [13, Lemma 6.1], we have

$$k_u(r) = o(r^2).$$

From this and Lemma 2.3, (2.4) is true for $p = 1$. The estimate ((1.3)) is given by Theorem 1.1 of [12].

In general, by considering $\widetilde{M} = M \times \mathbb{R}^4$, we can find $\widetilde{u}$ as above. By [2, p.458–460], $\widetilde{u}$ is independent of $y \in \mathbb{R}^4$ for $(x, y) \in M \times \mathbb{R}^4$. Let $u(x) = \widetilde{u}(x, y)$. Then $\Delta u = f$ and $k_u(r) = o(r^2)$. The existence of v is as in the previous case. $\square$

To apply Theorem 2.1 we need to construct a long time solution to the Hodge-Laplace heat equation. The rest of the paper is devoted to this.

3. AN INITIAL-BOUNDARY VALUE PROBLEM FOR $(1, 1)$-FORMS

We begin to construct η satisfying Theorem 2.1(a) by compact exhaustions. Hence we first consider the following initial-boundary value problem.

Let (M^n, g) be a complete noncompact Kähler manifold and let Ω be a bounded domain in M with smooth boundary. We want to discuss the initial-boundary value problem for real $(1, 1)$-forms

(3.1)
$$
\begin{cases}
\eta_t - \Delta \eta = 0, & \text{in } \Omega \times (0, T); \\
\lim_{t \to 0} \eta(x, t) = \rho(x), & x \in \Omega; \\
\mathbf{n}\, \eta = 0, & \text{on } \partial\Omega \times (0, T); \\
\mathbf{n}\, d\eta = 0, & \text{on } \partial\Omega \times (0, T).
\end{cases}
$$

where ρ is a smooth real $(1, 1)$-form. Here for a form ϕ, $\mathbf{n}\, \phi = \iota_\nu \phi$ is the normal part of ϕ, where ν is the unit outward normal, and ι_ν is the adjoint operator

of $\nu^* \wedge (\cdot)$; $\mathbf{t}\,\phi$ denotes the tangential part of ϕ. The readers are referred to [10] for details and the corresponding elliptic boundary value problem. $\Delta = -\Delta_d = -(d\delta + \delta d)$ which is the negative of the Hodge Laplacian on forms.

Let us first consider the underlining Riemannian structure of M and start with a simple lemma.

Lemma 3.1. *Let η be a two form on Ω (with real dimension m) and let x^i, $1 \leq i \leq m$ be local coordinates at the boundary near a point p such that $\frac{\partial}{\partial x^m}$ is the unit outward normal so that x^m is the signed distance from the boundary and x^α, $1 \leq \alpha \leq m-1$ are local coordinates of the boundary. Suppose $\eta = \sum_{i<j} \eta_{ij} dx^i \wedge dx^j$. Then at p the boundary conditions $\mathbf{n}\,\eta = 0$ and $\mathbf{n}\,d\eta = 0$ are equivalent to $\eta_{\alpha m} = 0$ and $\eta_{\alpha\beta,m} = \frac{\partial}{\partial x^m}\eta_{\alpha\beta} = 0$ for $1 \leq \alpha < \beta \leq m-1$.*

Proof. Then condition $\mathbf{n}\,\eta = 0$ means that $\eta_{\alpha m} = 0$ for $\alpha < m$. Then

$$(3.2)$$

$$
d\eta = \sum_{1 \leq i < j \leq m} \eta_{ij,k} dx^k \wedge dx^i \wedge dx^j
$$

$$
= \sum_{1 \leq \alpha < \beta \leq m-1} \eta_{\alpha\beta,k} dx^k \wedge dx^\alpha \wedge dx^\beta + \sum_{1 \leq \alpha \leq m-1} \eta_{\alpha m,k} dx^k \wedge dx^\alpha \wedge dx^m
$$

$$
= \sum_{\gamma=1}^{m-1} \sum_{1 \leq \alpha < \beta \leq m-1} \eta_{\alpha\beta,\gamma} dx^\gamma \wedge dx^\alpha \wedge dx^\beta + \sum_{1 \leq \alpha < \beta \leq m-1} \eta_{\alpha\beta,m} dx^m \wedge dx^\alpha \wedge dx^\beta
$$

$$
+ \sum_{\gamma=1}^{m-1} \sum_{1 \leq \alpha \leq m-1} \eta_{\alpha m,\gamma} dx^\gamma \wedge dx^\alpha \wedge dx^m.
$$

Here " $_,$ " means the partial derivative, in the second line the repeated index k was summed from 1 to m. Since $\eta_{\alpha m} = 0$ at the boundary, $\eta_{\alpha m,\gamma} = 0$ at the boundary. Combining this with $\mathbf{n}\,d\eta = 0$, we have $\eta_{\alpha\beta,m} = 0$ at the boundary for $1 \leq \alpha < \beta < m$.

Conversely, if $\eta_{\alpha m} = 0$ and $\eta_{\alpha\beta,m} = \frac{\partial}{\partial x^m}\eta_{\alpha\beta} = 0$ for $1 \leq \alpha < \beta \leq m-1$, it is easy to see from the above that $\mathbf{n}\,\eta = 0$ and $\mathbf{n}\,d\eta = 0$. $\square$

Recall the following basic fact [10, Lemma 7.5.3]. Let α, β be r and $r-1$ forms, then

$$(3.3) \qquad \int_\Omega \langle \alpha, d\beta \rangle = \int_\Omega \langle \delta\alpha, \beta \rangle + (-1)^{r-1} \int_{\partial\Omega} \langle \mathbf{n}\alpha, \mathbf{t}\beta \rangle.$$

Using this we have the following equation for $\eta(x,t)$, a solution to (3.1).

$$(3.4) \qquad \frac{d}{dt} \int_\Omega |\eta|^2 = 2 \int_\Omega \langle \Delta\eta, \eta \rangle = -2 \int_\Omega (|d\eta|^2 + |\delta\eta|^2)$$

where we have used the boundary conditions of η. In particular the L^2 norm of η is nonincreasing in t.

Let $P(x, y, t)$ be the fundamental solution of (3.1), whose existence is provided by [14, Proposition. 5.3] partially via Lemma 3.1 and (3.3). $P(x, y, t)$ is a double form with the property that if ρ is smooth real on $\overline{\Omega}$, then

$$(3.5) \qquad \eta(x, t) = (P(t)\rho)(x) = \int_\Omega P(x, y, t) \wedge *\rho(y)$$

is a solution of the initial-boundary value problem (3.1).

Lemma 3.2. *Let ρ be a smooth real form on $\overline{\Omega}$ and let η be given by (3.5). Then the following are true:*

(i) *For all $t > 0$,*

$$\int_\Omega |\eta|^2(x, t) \le \int_\Omega |\rho|^2(x).$$

(ii) *For any $T > 0$,*

$$\sup_{\Omega \times [0, T]} |\eta| \le C \sup_\Omega |\rho|.$$

(iii) *Suppose ρ has compact support in Ω, then η is continuous in $\overline{\Omega} \times [0, \infty)$.*

(iv) *Suppose ρ is closed, then $\eta(\cdot, t)$ is close for all $t > 0$.*

Proof. For (i), by approximate the initial data by forms with compact support so that we may assume the solution is continuous at $t = 0$ up to the corner $\partial \Omega \times \{0\}$. Then the result follows from (3.4).

For (ii) and (iii), it follow from the estimate on P, more precisely (5.4) of [14, Proposition 5.3]. By [14, Proposition 5.3], $d_x P(x, y, t) = \delta_y P(x, y, t)$ and $\mathbf{n}_y(P(x, y, t) = 0$, we have, by (3.3),

$$d\eta(x, t) = \int_\Omega \delta_y P(x, y, t) \wedge *\rho(y) = 0.$$

This proves (iv). $\qquad \square$

Lemma 3.3. *Let (M^n, g) be a complete noncompact Kähler manifold and let Ω be a bounded domain in M with smooth boundary. Let ρ be a smooth real $(1,1)$-form on $\overline{\Omega}$. Then there is a solution η of the initial-boundary value problem (3.1) such that η is a real $(1,1)$-form for all $t > 0$. If ρ is closed then η is also closed for $t > 0$. Moreover, η is smooth on $\overline{\Omega} \times [0, \infty)$ except at $\partial \Omega \times \{0\}$.*

Proof. Let $0 \le \phi_i \le 1$ be smooth functions with compact supports $\Omega_i \Subset \Omega$ so that $\Omega_i \uparrow \Omega$. Let $\rho^{(i)} = \phi_i \rho$ and let $\eta^{(i)}$ be the corresponding solutions with initial data $\rho^{(i)}$ given by (3.5). Obvious $\eta^{(i)}$ is real. To prove that $\eta^{(i)}$ is a $(1,1)$-form. First notice that Δ preserves $(1,1)$-forms. By Lemma 3.1, the boundary conditions satisfy the complementary conditions and compatibility condition at the corner $\partial \Omega \times \{0\}$ because $\rho^{(i)}$ has compact support. Hence we can solve (3.1) so that the solution is a $(1,1)$-form for $t > 0$, see [4, p.596] for example. Moreover both this solution and η are continuous up to $t = 0$ and

$\partial\Omega$. By (3.4), the two solutions are the same. Hence $\eta^{(i)}$ are $(1,1)$-forms for $t > 0$.

By Lemma 3.2(ii), and the Schauder estimates, we conclude that by passing to a subsequence, $\{\eta^{(i)}\}$ will converge to a solution η of (3.1), which is a real $(1,1)$-form and smooth except at the corner $\partial\Omega \times \{0\}$. Moreover η is also given by the representation formula (3.5) by Lemma 3.2(i). If ρ is closed then η is closed by Lemma 3.2(iv). $\qquad\square$

The following has been proved in [1]. For the sake of completeness, we sketch the computations.

Lemma 3.4. *Let η be a 2-form satisfying the boundary conditions of (3.1).*

$$\frac{\partial}{\partial\nu}||\eta||^2 = -\sum_{\alpha,\beta,\gamma=1}^{n-1} 4h_{\alpha\beta}\eta_{\alpha\gamma}\eta_{\beta\gamma}.$$

Here $(h_{\alpha\beta})$ is the second fundamental form with the outward unit normal.

Proof. At a point p at the boundary, we introduce coordinate system as in the proof of Lemma 3.1. We may also assume that $g_{ij} = \delta_{ij}$ at p. Here and below, i, j etc are from 1 to m and α, β are from 1 to $m-1$. Then $\eta_{\alpha m} = 0$ at p.

Now consider η as a two tensor η_{ij} such that $\eta_{ij} = -\eta_{ji}$. Then $||\eta||^2 = g^{ik}g^{jl}\eta_{ij}\eta_{kl}$, and at $p \in \partial\Omega$

$$(3.6) \quad \begin{aligned} \frac{\partial}{\partial x_m}||\eta||^2 &= \frac{\partial}{\partial x_m}\left(g^{ik}g^{jl}\right)\eta_{ij}\eta_{kl} + g^{ik}g^{jl}\eta_{ij,m}\eta_{kl} + g^{ik}g^{jl}\eta_{ij}\eta_{kl,m} \\ &= \frac{\partial}{\partial x_m}\left(g^{ik}\right)g^{jl}\eta_{ij}\eta_{kl} + g^{ik}\frac{\partial}{\partial x_m}\left(g^{jl}\right)\eta_{ij}\eta_{kl} + 2\eta_{ij}\eta_{ij,m} \\ &= -g_{ik,m}\eta_{ij}\eta_{kj} - g_{jl,m}\eta_{ij}\eta_{il} \\ &= -2g_{\alpha\beta,m}\eta_{\alpha\gamma}\eta_{\beta\gamma}. \end{aligned}$$

Since $g_{\alpha\beta,m} = 2h_{\alpha\beta}$, the second fundamental form with respect to the unit outward normal, we have

$$\frac{\partial}{\partial x_m}||\eta||^2 = -4h_{\alpha\beta}\eta_{\alpha\gamma}\eta_{\beta\gamma}$$

which is the claim of the lemma. $\qquad\square$

Now consider η a real $(1,1)$-form in a Kähler manifold (M^n, g) where n is the complex dimension and Ω is a bounded domain with smooth boundary. We want to understand the above lemma for this special case. Assume that η satisfies the boundary conditions in (3.1). Let x^i be local coordinates near a point p at the boundary as above and $\frac{\partial}{\partial x^{2n}}$ is the unit outward normal. Assume near boundary point p, we have a holomorphic coordinates z^i, such that at p, $\frac{\partial}{\partial z^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} - \sqrt{-1}\frac{\partial}{\partial x^{i+n}}\right)$, $\frac{\partial}{\partial\bar{z}^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} + \sqrt{-1}\frac{\partial}{\partial x^{i+n}}\right)$ so that $J\frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^{i+n}}$, $1 \leq i \leq n$. Here J is the underlining complex structure. Suppose $\eta = \eta_{i\bar{j}}dz^i \wedge$

$d\bar{z}^j$ with $\overline{\eta_{i\bar{j}}} = \eta_{j\bar{i}}$. Then at p the boundary conditions $\mathbf{n}\eta = 0$ and $\mathbf{n}d\eta = 0$ are equivalent to $\eta_{\alpha m} = 0$ and $\eta_{\alpha\beta,m} = \frac{\partial}{\partial x^m}\eta_{\alpha\beta} = 0$ for $1 \le \alpha < \beta \le m - 1$.

Let the corresponding real form be $\lambda = \sqrt{-1}\eta$, which can be written as $\sum_{s,t=1}^{2n} \lambda_{st} dx^s \wedge dx^t$. By our choice of z^i, $dz^i = dx^i + \sqrt{-1}dx^{i+n}$, etc, and at p

$$
\begin{aligned}
2\lambda =& \sqrt{-1}\eta - \sqrt{-1}\bar{\eta} \\
=& \sqrt{-1}\eta_{i\bar{j}}\left(dx^i + \sqrt{-1}dx^{i+n}\right) \wedge \left(dx^j - \sqrt{-1}dx^{j+n}\right) \\
& - \sqrt{-1}\overline{\eta_{i\bar{j}}}\left(dx^i - \sqrt{-1}dx^{i+n}\right) \wedge \left(dx^j + \sqrt{-1}dx^{j+n}\right) \\
=& \sqrt{-1}(\eta_{i\bar{j}} - \overline{\eta_{i\bar{j}}})\left(dx^i \wedge dx^j + dx^{i+n} \wedge dx^{j+n}\right) \\
& + (\eta_{i\bar{j}} + \overline{\eta_{i\bar{j}}})\left(dx^i \wedge dx^{j+n} + dx^j \wedge dx^{i+n}\right).
\end{aligned}
$$
(3.7)

Hence for $1 \le i, j \le n$

$$2\lambda_{ij} = 2\lambda_{i+n\,j+n} = \sqrt{-1}(\eta_{i\bar{j}} - \eta_{j\bar{i}}), \quad 2\lambda_{i\,j+n} = 2\lambda_{j\,i+n} = (\eta_{i\bar{j}} + \eta_{j\bar{i}}).$$

In the following, α, β, i, j etc will range from 1 to $n-1$, and $m = 2n$. By Lemma 3.4

$$\frac{\partial}{\partial\nu}||\eta||^2 = -4\sum_{s,t,q=1}^{2n-1} h_{st}\lambda_{sq}\lambda_{tq}.$$

where $h_{st} = h(\frac{\partial}{\partial x^s}, \frac{\partial}{\partial x^t})$. Now $\lambda_{i+n\,m} = 0$ for $1 \le i \le n-1$. Hence $\lambda_{i\,n} = \lambda_{i+n\,m} = 0$, and $\lambda_{i+n\,n} = \lambda_{i\,m} = 0$ for $1 \le i \le n-1$, which then implies

$$\frac{\partial}{\partial\nu}||\eta||^2 = -4\sum_{s,t,q=1,s\neq n,t\neq n,q\neq n}^{2n-1} h_{st}\lambda_{sq}\lambda_{tq}.$$

With the above and the Kähler condition, we have the the following one.

Lemma 3.5. *Let η be any real $(1,1)$-form satisfying the boundary conditions in (3.1), and let (M,g) be a Kähler manifold.*

(a) *On the boundary $\partial\Omega$, if $k_{\alpha\bar{\beta}} = h(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial\bar{z}^\beta})$,*

$$\frac{\partial}{\partial\nu}||\eta||^2 = -16\sum_{\alpha,\beta,\gamma=1}^{n-1} k_{\alpha\bar{\beta}}\eta_{\alpha\bar{\gamma}}\eta_{\gamma\bar{\beta}}.$$

Particularly, $\frac{\partial}{\partial\nu}||\eta||^2 \le 0$ if $\partial\Omega$ is pseudo-convex.

(b) *Suppose that η is closed. Let $v = \mathrm{tr}(\eta)$. Then $\frac{\partial v}{\partial\nu} = 0$ at $\partial\Omega$.*

Proof. (a) This may be well-known. We include a proof for the sake of completeness. First since

$$
\begin{aligned}
h_{i+n\,j+n} =& h\left(J\frac{\partial}{\partial x^i}, J\frac{\partial}{\partial x^j}\right) = h\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = h_{ij}, \\
h_{i\,j+n} =& h\left(\frac{\partial}{\partial x^i}, J\frac{\partial}{\partial x^j}\right) = -h\left(J\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = -h_{i+n\,j},
\end{aligned}
$$
(3.8)

$$(3.9) \qquad h\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right) = \frac{1}{4}\left(h_{ij} - h_{i+n\,j+n}\right) - \sqrt{-1}\left(h_{ij+n} + h_{j\,i+n}\right) = 0.$$

Moreover, for $1 \le \alpha, \beta \le n$, $h_{\alpha\beta} = k_{\alpha\bar\beta} + k_{\beta\bar\alpha} = h_{\alpha+n\,\beta+n}$ and $h_{\alpha\,\beta+n} = -\sqrt{-1}(k_{\alpha\bar\beta} - k_{\beta\bar\alpha})$. Noting that $\overline{k_{\alpha\bar\beta}} = k_{\beta\bar\alpha}$, thus we have

$$-\frac{1}{4}\frac{\partial}{\partial\nu}||\eta||^2 = \sum_{s,t,q=1, s\neq n, t\neq n, q\neq n}^{2n-1} h_{st}\lambda_{sq}\lambda_{tq}$$

$$(3.10) \qquad = \sum_{q\neq n, 2n}\sum_{\alpha,\beta=1}^{n-1}\left(k_{\alpha\bar\beta} + k_{\beta\bar\alpha}\right)\left(\lambda_{\alpha q}\lambda_{\beta q} + \lambda_{\alpha+n\,q}\lambda_{\beta+n\,q}\right)$$

$$- 2\sqrt{-1}\sum_{q\neq n, 2n}\sum_{\alpha\,\beta=1}^{n-1}(k_{\alpha\bar\beta} - k_{\beta\bar\alpha})\lambda_{\alpha q}\lambda_{\beta+n\,q}$$

$$= I + II.$$

Now choose $\{z^\alpha\}$ such that $k_{\alpha\bar\beta} = a_\alpha\delta_{\alpha\beta}$. This can be done because the point p, k is a Hermitian form on the subspace of $T_p^{(1,0)}(M)$ which is orthogonal to the complex line spanned by $\nu + \sqrt{-1}J\nu$. Because $\lambda_{\alpha+n\,\gamma+n} = \lambda_{\alpha\gamma}$ and $\lambda_{\alpha\,\gamma+n} = \lambda_{\gamma\,\alpha+n} = -\lambda_{a+n\,\gamma}$ etc,

$$I = 4\sum_{\alpha,\gamma=1}^{n-1} a_\alpha\left(\lambda_{\alpha\gamma}^2 + \lambda_{\alpha\,\gamma+n}^2\right)$$

$$(3.11) \qquad = \sum_{\alpha,\gamma=1}^{n-1} a_\alpha\left(-(\eta_{\alpha\bar\gamma} - \eta_{\gamma\bar\alpha})^2 + (\eta_{\alpha\bar\gamma} + \eta_{\gamma\bar\alpha})^2\right)$$

$$= 4\sum_{\alpha,\gamma=1}^{n-1} a_\alpha\eta_{\alpha\bar\gamma}\eta_{\gamma\bar\alpha}.$$

On the other hand,

$$II = -2\sqrt{-1}\sum_{\alpha,\beta,\gamma=1}^{n-1}(k_{\alpha\bar\beta} - k_{\beta\bar\alpha})\left(\lambda_{\alpha\gamma}\lambda_{\beta+n\,\gamma} + \lambda_{\alpha\,\gamma+n}\lambda_{\beta+n\,\gamma+n}\right)$$

$$(3.12) \qquad = -2\sqrt{-1}\sum_{\alpha\,\beta,\gamma=1}^{n-1}(k_{\alpha\bar\beta} - k_{\beta\bar\alpha})\left(\lambda_{\alpha\gamma}\lambda_{\beta+n\,\gamma} + \lambda_{\gamma\,\alpha+n}\lambda_{\beta\gamma}\right)$$

$$= 0.$$

(b) By (8.1.19) of [10], which asserts that

$$\iota_\nu\bar\partial^*\eta = 0,$$

the closeness of η implies that,

$$\iota_\nu\partial v = \iota_\nu\bar\partial v = 0,$$

by the identities $\partial\Lambda - \Lambda\partial = -\sqrt{-1}\bar\partial^*$, $\bar\partial\Lambda - \Lambda\bar\partial = \sqrt{-1}\partial^*$. $\qquad\square$

4. Global solutions to the Hogde-Laplace heat equation 1

Recall that a Kähler manifold is said to have *nonnegative orthogonal bisectional curvature* if at any point $R_{i\bar{i}j\bar{j}} \geq 0$ for all unitary pair $\{e_i, e_j\}$.

Theorem 4.1. *Let (M, g) be a complete Kähler manifold with nonnegative orthogonal bisectional curvature and with nonnegative Ricci curvature. Assume that ρ is a d-closed $(1, 1)$-form such that $f = \|\rho\|$ satisfying*

$$(4.1) \qquad \limsup_{R \to \infty} \frac{k_f(R)}{R^2} = 0,$$

and there exists a constant $a \geq 0$ such that $\rho + a\omega \geq 0$, where ω is the Kähler form of M. Then there exists a solution of

$$(4.2) \qquad \begin{cases} \eta_t - \Delta\eta &= 0, \ \text{in } M \times [0, \infty); \\ \eta(x, 0) &= \rho(x), \quad x \in M. \end{cases}$$

such that η is a closed $(1, 1)$-form and is nonnegative provided that ρ is nonnegative. Furthermore,

(a) *for any $T > 0$,*

$$(4.3) \qquad \lim_{R \to \infty} \frac{1}{R^2 V_o(R)} \int_0^T \int_{B_o(R)} \|\eta\|(x, t) dx dt = 0;$$

(b) $\lim_{t \to \infty} \eta(x, t) = 0$ *for all $x \in M$ provided that $\lim_{R \to \infty} k_f(R) \to 0$.*

For the proof, we need some lemmas. Let M be a complete noncompact Riemannian manifold with non-negative Ricci curvature and let $h(x) \geq 0$ and $v(x, t)$ be functions as in Lemma 2.3. Then v solves the initial value problem

$$(4.4) \qquad \begin{cases} v_t - \Delta v = 0; \\ v(x, 0) = h(x). \end{cases}$$

The function $v(x, t)$ shall serve as a global barrier. But first we establish the relation between the local and global barriers.

Consider a compact exhaustion $\{\Omega_i\}$ with smooth boundary such that there exists $R_i \to \infty$ with $B_o(R_i) \subset \Omega_i \subset B_o(2R_i)$. Let $\phi^{(i)}$ be the solution of

$$(4.5) \qquad \begin{cases} u_t - \Delta u = 0, \quad \text{in } \Omega_i; \\ \frac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial\Omega_i \times (0, \infty); \\ u(x, 0) = h(x). \end{cases}$$

The following result relates v and $\phi^{(i)}$.

Lemma 4.1. *Suppose h satisfies (5.2). After possibly passing to a subsequence, $\phi^{(i)}$ converge to v on compact sets in $M \times [0, \infty)$.*

Proof. Let us first assume that $\phi^{(i)}$ is continuous in $\overline{\Omega}_i \times [0, \infty)$. Then by the maximum principle, $\phi^{(i)} \geq 0$. On the other hand, by the Neumann boundary condition, we have

$$\frac{d}{dt} \int_{\Omega_i} \phi^{(i)} = 0.$$

Hence

(4.6)
$$\int_{\Omega_i} \phi^{(i)}(x,t)\,dx = \int_{\Omega_i} h(x)\,dx.$$

As in the proof of Lemma 3.3, we can remove the assumption on the continuity of $\phi^{(i)}$ so that (4.6) is still true. Now

$$\left(\Delta - \frac{\partial}{\partial t}\right)(v - \phi^{(i)})^2 \geq 0.$$

Since that at $t = 0$, $v - v^{(i)} = 0$, by [6, Theorem 1.2], we have for any $T > 0$

$$\sup_{B_o(\frac{1}{4}R_i)\times[0,T]} |v - \phi^{(i)}| \leq \frac{C_1}{R_i^2 V_o(R_i)} \int_0^T \int_{B_o(R_i)} |v - \phi^{(i)}|\,(x,t)\,dx\,dt$$

$$\leq \frac{C_1}{R_i^2 V_o(R_i)} \int_0^T \left(\int_{B_o(2R_i)} (|v| + h)(x)\,dx\right) dt.$$

for some constant C_1 independent of i, where we have used the volume comparison, (4.6) and the fact $B_o(R_i) \subset \Omega_i \subset B_o(2R_i)$.

By the assumption (5.2), and the volume comparison, we have

$$\frac{1}{V_o(R)} \int_{B_o(2R)} h(y)\,dy = o(R^2)$$

as $R \to \infty$. By Lemma 2.3, we conclude that

(4.7)
$$\sup_{B_o(\frac{1}{4}R_i)\times[0,T]} |v - \phi^{(i)}| \leq C_i$$

where $C_i \to 0$ as $i \to \infty$. From this the result follows. $\square$

We also need the following:

Lemma 4.2. *Let (M,g) be a complete noncompact Kähler manifold with non-negative orthogonal bisectional curvature. Let $\eta(x,t)$ be the solution initial-boundary value problem (3.1) provided by Lemma 3.2. If $\rho \geq 0$ then $\eta(x,t) \geq 0$ for $t > 0$.*

Proof. The proof of Lemma 3.3 shows that there are two approaches to obtain $\eta(x,t)$. One is via the representation formula (3.5). The other is via the limit of a sequence of solutions $\eta^{(i)}(x,t)$ which solves (3.1) but with smooth compatible initial condition $\phi_i\rho$ with ϕ_i being cut-off functions. Hence it suffices to prove that $\eta^{(i)}(x,t) \geq 0$. Now the argument in the proof of Proposition 3.1 of [11] can be applied to obtain this since $\eta^{(i)}$ has the sufficient regularity. For the sake of the completeness we include the argument below using the notations from the previous section.

By the general maximum principle, Theorem 2.1 in [11], it suffices to check that (i) if at $p \in \Omega$, for some normal coordinate (z^i), $\eta_{1\bar{1}} < 0$ and $\eta_{i\bar{j}}a^i\overline{a_j} \geq \eta_{1\bar{1}}$

for all a with $|a| = 1$, $\mathcal{KB}(\eta)_{1\bar{1}} \geq 0$, where

$$\mathcal{KB}(\eta)_{i\bar{j}} = R_{i\bar{j}k\bar{l}}\eta_{\lambda\bar{k}} - \frac{1}{2}R_{i\bar{k}}\eta_{k\bar{j}} - \frac{1}{2}R_{k\bar{j}}\eta_{i\bar{k}}$$

defined in terms of any unitary frame; (ii) if $p \in \partial\Omega$, and for some local orthogonal real frame $(\frac{\partial}{\partial x^i})$, $1 \leq i \leq m = 2n$, with corresponding complex frame $(\frac{\partial}{\partial z^i})$, $1 \leq i \leq n$ with $\frac{\partial}{\partial z^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} - \sqrt{-1}\frac{\partial}{\partial x^{i+n}}\right)$ and $\frac{\partial}{\partial x^m} = \nu$, as in the discussion of the previous section before Lemma 3.5, with $\eta_{X\bar{X}} < 0$ and $\eta_{i\bar{j}}a^i\overline{a_j} \geq \eta_{X\bar{X}}$ for some $X \in T'_pM$ and all $a \in \mathbb{C}^n$ with $|X| = |a| = 1$, we have that $\frac{\partial}{\partial x^m}\eta_{X\bar{X}} = 0$.

To check (i), let $\widetilde{\eta}_{i\bar{j}} = \eta_{i\bar{j}} - \eta_{1\bar{1}}g_{i\bar{j}}$. The assumption $\eta_{i\bar{j}}a^i\overline{a_j} \geq \eta_{1\bar{1}}$ for all a with $|a| = 1$ implies that $(\widetilde{\eta}_{i\bar{j}}) \geq 0$ as a Hermitian symmetric form. Now it is easy to check that $\mathcal{KB}(\eta) = \mathcal{KB}(\widetilde{\eta})$ and $(\mathcal{KB}(\widetilde{\eta})_{i\bar{j}}) \geq 0$. For the last claim one can see, for example the proof of (2.14) in [13]. (If using the the first variation on the fact that $X = e_1$ being the minimizing unit vector, the claim is equivalent to $\sum_{k,l\geq2} R_{1\bar{1}k\bar{l}}\widetilde{\eta}_{l\bar{k}} \geq 0$, which clearly holds under the nonnegativity assumption of the orthogonal bisectional curvature.)

To check (ii), first observe that if (λ_{st}) denotes the real symmetric form associated with η (as in the discussion of the previous section), using

$$2\lambda_{ij} = 2\lambda_{i+n\,j+n} = \sqrt{-1}(\eta_{i\bar{j}} - \eta_{j\bar{i}}), \quad 2\lambda_{i\,j+n} = 2\lambda_{j\,i+n} = (\eta_{i\bar{j}} + \eta_{j\bar{i}})$$

for $1 \leq i, j \leq n$, and $\lambda_{sm} = 0$ by Lemma 3.1, we have that $\eta_{i\bar{n}} = \eta_{n\bar{i}} = 0$. Now making use of the assumption that X is the minimizing unit vector, define the functional, for any small complex number ϵ,

$$\mathcal{I}(\epsilon) \doteq \frac{\eta_{(X+\epsilon Z)\overline{X+\epsilon Z}}}{|X + \epsilon Z|^2}.$$

The first variation $\frac{\partial}{\partial\epsilon}\mathcal{I}(0) = \frac{\partial}{\partial\bar{\epsilon}}\mathcal{I}(0) = 0$ then implies that

$$\eta_{X\overline{Z}} - \eta_{X\overline{X}}\langle X, Z \rangle = 0$$

for any Z with $\langle\cdot,\cdot\rangle$ being the Hermitian product. By letting $Z = \frac{\partial}{\partial z^n}$, Lemma 3.1 implies that $\langle X, \frac{\partial}{\partial z^n}\rangle = 0$. Now the claim $\frac{\partial}{\partial x^m}\eta_{X\bar{X}} = 0$ again follows from Lemma 3.1. □

We are ready to prove Theorem 4.1.

Proof of Theorem 4.1. Suppose $\rho + a\omega$ is nonnegative for some $a \geq 0$. Then we can find bounded domains Ω_i with smooth boundary such that $B_o(R_i) \subset \Omega_i \subset B_o(2R_i)$. By Lemma 3.3, we can find $\eta^{(i)}$ which is a solution to (3.1) on Ω_i with initial data $\rho + a\omega$ and by Lemma 4.2, $\eta^{(i)}$ is nonnegative. The trace of $\eta^{(i)}$ satisfies the heat equation with initial value $\phi^{(i)} = \text{tr}(\rho + a\omega)$ and has Neumann boundary condition by Lemma 3.5(b). Note that the norm of $\rho + a\omega$ satisfies (5.2) if $||\rho||$ satisfies the condition. By Lemma 4.1 the trace of

$\eta^{(i)}$ converges in compact sets to v with

$$v(x,t) = \int_M H(x,y,t)\mathrm{tr}(\rho + a\omega)(y)dy.$$

Since $\eta^{(i)}$ is nonnegative, $||\eta^{(i)}|| \leq C(n)\phi^{(i)}$. We can conclude that $\eta^{(i)} \to \widetilde{\eta}$ which is a solution to (4.2), but with initial data $\rho + a\omega$ and satisfying (4.3). Let $\eta = \widetilde{\eta} - a\omega$. It is easy to see that η also solves (4.2) and satisfies (4.3).

On the other hand, solve the Dirichlet boundary problem on Ω_i:

(4.8)
$$\begin{cases} \xi_t^{(i)} - \Delta\xi^{(i)} = 0, \text{ in } \Omega_i \times (0,\infty); \\ \xi^{(i)}(x,0) = \rho(x), \ x \in \Omega_i; \\ \xi^{(i)}(x,t) = 0, \text{ on } \partial\Omega_i \times (0,\infty). \end{cases}$$

For the solution $\xi^{(i)}$, via the maximum principle it is easy to see that

$$\|\xi^{(i)}\| \leq \int_M H(x,y,t)\|\rho\|(y)dy,$$

By the estimate in Lemma 2.3, passing to a subsequence $\xi^{(i)}$ converges to a limit ξ solving (4.2) and satisfies (4.3).

Now apply Lemma 2.2 to $\|\eta - \xi\|$ and conclude that $\eta = \xi$. Hence $\|\eta\|(x,t) \leq u(x,t)$. Then the estimate (4.3), and the last claim (if $k_f(R) \to 0$) also follows from Lemma 2.3 and [6, Theorem 1.1]. $\qquad\square$

5. Global solutions to the Hodge-Laplace heat equation 2

In this section, we consider the case that ρ may not be bounded from below. A Kähler manifold (M^n, g) is said to satisfy *nonnegative quadratic orthogonal bisectional curvature* if at any point, for any unitary frame e_i, and any real numbers a_i,

(5.1)
$$\sum_{i,j} R_{i\bar{i}j\bar{j}}(a_i - a_j)^2 \geq 0.$$

We want to prove the following:

Theorem 5.1. *Let (M, g) be a complete Kähler manifold with nonnegative quadratic orthogonal bisectional curvature and with nonnegative Ricci curvature. Assume that ρ is a d-closed $(1,1)$-form such that $f = \|\rho\|$ satisfying*

(5.2)
$$\limsup_{R \to \infty} \frac{k_f(R)}{R^2} = 0.$$

Suppose there exist $R_i \to \infty$ and pseudo convex domains Ω_i such that $B_o(R_i) \subset \Omega_i \subset B_o(2R_i)$ with smooth boundary. Then there exists a solution of

(5.3)
$$\begin{cases} \eta_t - \Delta\eta = 0, \ in \ M \times [0,\infty); \\ \eta(x,0) = \rho(x), \quad x \in M. \end{cases}$$

such that η is a closed $(1,1)$-form. Furthermore, the conclusions (a) and (b) of Theorem 4.1 hold.

We need the following:

Lemma 5.1. *Let (M^n, g) be a complete noncompact Kähler manifold with nonnegative quadratic orthogonal bisectional curvature and let Ω be a bounded domain in M with smooth boundary. Let ρ be a smooth real $(1,1)$-form on $\overline{\Omega}$. Let η be the solution of (3.1) in Lemma 3.3. Let ϕ be the solution of the initial value problem*

$$(5.4) \qquad \begin{cases} \phi_t - \Delta\phi = 0, & in\ \Omega; \\ \frac{\partial\phi}{\partial\nu} = 0, & on\ \partial\Omega \times (0, \infty); \\ \phi(x, 0) = ||\rho||. \end{cases}$$

If $\partial\Omega$ is pseudo convex, then $||\eta||(x, t) \le u(x, t)$ for all $(x, t) \in \Omega \times (0, \infty)$.

Proof. To prove the lemma, we may assume that ρ has compact support. Then η and ϕ are smooth up to the corner. Now by [8] and the curvature condition (see [3, p. 229–230]), we have for any $\epsilon > 0$,

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left(||\eta||^2 + \epsilon\right)^{\frac{1}{2}} \le 0.$$

The result follows by the maximum principle, pseudo-convexity, Lemma 3.5, by comparing $(||\eta||^2 + \epsilon)^{\frac{1}{2}}$ with $u + \epsilon$. $\qquad\square$

Proof of Theorem 5.1. By Lemma 3.3 for each Ω_i we can find a solution $\eta^{(i)}$ of (3.1) on $\Omega_i \times (0, \infty)$ which is smooth and closed except at the corner. Moreover, by Lemma 5.1 and (4.7), we conclude that for all $T > 0$

$$(5.5) \qquad \sup_{B_o(\frac{1}{4}R_i) \times [0,T]} ||\eta^{(i)}|| \le u + C_i$$

where $C_i \to 0$ as $i \to \infty$ and u is the solution of (4.4). Passing to a subsequence, we conclude that $\{\eta^{(i)}\}$ converges to a solution η of (4.2). Moreover $||\eta||(x, t) \le u(x, t)$.

Now by (5.2) we have that $\frac{k_f(4R)}{R^2} \to 0$ as $R \to \infty$. By Lemma 2.3 and the fact that $||\eta||(x, t) \le u(x, t)$, we conclude that (4.3) is true for any $T > 0$.

By [6, Theorem 1.1], if additionally assume that $k_f(R) \to 0$, we have that $u(x, t) \to 0$ as $t \to \infty$. Hence $\eta(x, t) \to 0$ as $t \to \infty$. $\qquad\square$

6. Global solutions to the Hodge-Laplace heat equation 3

Under an additional assumption on the curvature, we may remove the condition on the existence of pseudo convex exhaustion in Theorem 5.1. We begin with a lemma.

Lemma 6.1. *Let (M^n, g) be a Kähler manifold and σ is a (2,1)-form. In a unitary frame,*

$$(6.1) \qquad \begin{aligned} \Delta||\sigma||^2 =& 2\langle\Delta\sigma, \sigma\rangle + 2||\nabla\sigma||^2 + 2\left(R_{i\bar{l}}g_{m\bar{k}} - R_{i\bar{k}m\bar{l}}\right)\sigma_{lj\bar{m}}\overline{\sigma_{ij\bar{k}}} \\ & + 2\left(R_{j\bar{l}}g_{m\bar{k}} - R_{j\bar{k}m\bar{l}}\right)\sigma_{il\bar{m}}\overline{\sigma_{ij\bar{k}}} + 2R_{l\bar{k}}\sigma_{ij\bar{l}}\overline{\sigma_{ij\bar{k}}} \end{aligned}$$

Proof. For $\sigma = \sigma_{ij\bar{k}}dz^i \wedge dz^j \wedge d\bar{z}^k$, we have

$$(6.2) \qquad\qquad -\Delta_d\sigma = g^{i\bar{j}}\left(\nabla_i\nabla_{\bar{j}} + \nabla_{\bar{i}}\nabla_j\right)\sigma + E(\sigma)$$

where

$$(6.3) \qquad (E(\sigma))_{ij\bar{k}} = -R_i^l\sigma_{lj\bar{k}} - R_j^l\sigma_{il\bar{k}} - R_{\bar{k}}^{\bar{l}}\sigma_{ij\bar{l}} + 2R_i{}^{l\bar{m}}{}_{\bar{k}}\,\sigma_{lj\bar{m}} + 2R_j{}^{l\bar{m}}{}_{\bar{k}}\,\sigma_{il\bar{m}}.$$

Therefore,

$$(6.4)$$
$$\begin{aligned}
\Delta\|\sigma\|^2 = & -2\langle\Delta_d\sigma,\sigma\rangle + 2\|\nabla\sigma\|^2 + 2R_{i\bar{l}}\sigma_{lj\bar{k}}\overline{\sigma_{ij\bar{k}}} + 2R_{j\bar{l}}\sigma_{il\bar{k}}\overline{\sigma_{ij\bar{k}}} + 2R_{l\bar{k}}\sigma_{ij\bar{l}}\overline{\sigma_{ij\bar{k}}} \\
& -2R_{i\bar{l}m\bar{k}}\sigma_{lj\bar{m}}\overline{\sigma_{ij\bar{k}}} - 2R_{j\bar{l}m\bar{k}}\sigma_{il\bar{m}}\overline{\sigma_{ij\bar{k}}} \\
= & -2\langle\Delta_d\sigma,\sigma\rangle + 2\|\nabla\sigma\|^2 + 2R_{i\bar{l}}\sigma_{lj\bar{k}}\overline{\sigma_{ij\bar{k}}} + 2R_{j\bar{l}}\sigma_{il\bar{k}}\overline{\sigma_{ij\bar{k}}} + 2R_{l\bar{k}}\sigma_{ij\bar{l}}\overline{\sigma_{ij\bar{k}}} \\
& -2R_{i\bar{k}m\bar{l}}\sigma_{lj\bar{m}}\overline{\sigma_{ij\bar{k}}} - 2R_{j\bar{k}m\bar{l}}\sigma_{il\bar{m}}\overline{\sigma_{ij\bar{k}}} \\
= & -2\langle\Delta_d\sigma,\sigma\rangle + 2\|\nabla\sigma\|^2 + 2\left(R_{i\bar{l}}g_{m\bar{k}} - R_{i\bar{k}m\bar{l}}\right)\sigma_{lj\bar{m}}\overline{\sigma_{ij\bar{k}}} \\
& +2\left(R_{j\bar{l}}g_{m\bar{k}} - R_{j\bar{k}m\bar{l}}\right)\sigma_{il\bar{m}}\overline{\sigma_{ij\bar{k}}} + 2R_{l\bar{k}}\sigma_{ij\bar{l}}\overline{\sigma_{ij\bar{k}}}
\end{aligned}$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 6.1. *Assume that (M,g) is as in Theorem 5.1 except that Ω_s is pseudo-convex. Instead, suppose that there exists $a \in \mathbb{R}$ such that*

$$(6.5) \qquad\qquad \left(R_{i\bar{l}}g_{m\bar{k}} - R_{i\bar{k}m\bar{l}}\right)\xi_{l\bar{m}}\overline{\xi_{i\bar{k}}} \geq -a^2\sum_{i,k}|\xi_{i\bar{k}}|^2$$

for all matrix $(\xi_{i\bar{k}})$ and $\int_0^\infty k_f(s)ds < \infty$. Then there is a solution η of (5.3) which is closed. Moreover η satisfies (2.2) for $p=1$ and $\lim_{t\to\infty}\eta(x,t) = 0$.

Proof. Let $\phi^{(i)}$ be a smooth cutoff function so that $0 \leq \phi^{(i)} \leq 1$, $\phi^{(i)} = 1$ on $B_o(R_i)$ and $\phi^{(i)} = 0$ outside $B_o(2R_i)$ and $|\nabla\phi^{(i)}| \leq C_1/R_i$ where $R_i \to \infty$ and C_1 is independent of i. Now for fixed i, let $\rho^{(i)} = \phi^{(i)}\rho$. Let Ω_s be compact exhaustion and let $\xi^{(s)}$ be the solution of

$$(6.6) \qquad \begin{cases} \xi_t^{(s)} - \Delta\xi^{(s)} = 0, \text{ in } \Omega_s \times (0,\infty); \\ \lim_{t\to 0}\xi^{(s)}(x,t) = \rho^{(i)}(x), \ x \in \Omega_s; \\ \mathbf{n}\xi^{(s)} = 0, \text{ on } \partial\Omega_s \times (0,\infty); \\ \mathbf{n}d\xi^{(s)} = 0, \text{ on } \partial\Omega_s \times (0,\infty) \end{cases}$$

given by Lemma 3.3. We have

$$\int_{\Omega_s}\|\xi^{(s)}\|^2(x,t) \leq \int_{\Omega_s}\|\rho^{(s)}\|^2 \leq C_2$$

where C_2 is independent of s because $\rho^{(i)}$ has compact support. Moreover, $d\xi^{(s)}$ (with initial value $d\rho^{(i)}$) also satisfies the Hodge-Laplace heat equation with $\mathbf{n}d\xi^{(s)} = 0$ and $\mathbf{n}d(d\xi^{(s)}) = 0$. Since $\rho^{(i)}$ has compact support, so $\int_{\Omega_s}\|d\xi^{(s)}\|^2$,

is continuous at $t = 0$ if s is large enough because the compatibility conditions hold at the corner. Hence

$$\int_{\Omega_s} ||d\xi^{(s)}||^2 \le \int_{\Omega_s} ||d\rho^{(i)}||^2.$$

On the other hand, $||\xi^{(s)}||$ is a subsolution of the heat equation because M has nonnegative quadratic orthogonal bisectional curvature. By [6, Theorem 1.2], for any $T > 0$ and $R > 0$, there is a constant C_2 which is independent of R, T, and s such that

$$\sup_{B_o(R) \times [0,T]} ||\xi^{(s)}|| \le C_2$$

provided R is large enough. Hence passing to a subsequence if necessary, we conclude that $\xi^{(s)}$ converge to a solution $\eta^{(i)}$ of (5.3). Moreover, $\eta^{(i)}$ is bounded on M. Since $||\eta^{(i)}||$ is a bounded subsolution of the heat equation with initial value $||\rho^{(i)}||$, we have

$$||\eta^{(i)}||(x,t) \le \int_M H(x,y,t)||\rho^{(i)}||(y)dy \le \int_M H(x,y,t)||\rho||(y)dy.$$

Moreover, since $\eta^{(i)} = \lim_{s\to\infty} \xi^{(s)}$

$$(6.7) \qquad \int_M ||d\eta^{(i)}||^2 \le \int_M ||d\rho^{(i)}||^2.$$

Passing to a subsequence, $\eta^{(i)}$ will converge uniformly on compact sets of $M \times [0,\infty)$ to a solution η of (5.3) with initial value ρ such that

$$||\eta||(x,t) \le \int_M H(x,y,t)||\rho||(y)dy.$$

From this it is easy to conclude the decay asserts of Theorem 5.1.

It remains to prove that η is closed. By Lemma 6.1 and the fact that Ricci is nonnegative, we have

$$\left(\Delta - \frac{\partial}{\partial t}\right)||d\eta^{(i)}|| \ge -C_3||d\eta^{(i)}||.$$

for some constant C_3 independent of i and t in the weak sense. Hence $e^{-C_3t}||d\eta^{(i)}||$ is a subsolution of the heat equation. Moreover, since $d\eta^{(i)}$ also satisfies the Hodge-Laplace heat equation and (6.7) it implies that $||d\eta^{(i)}||$ is bounded on M. Hence

$$e^{-Ct}||d\eta^{(i)}||(x,t) \le \int_M H(x,y,t)||d\rho^{(i)}||(y)\,dy = w_i(x,t).$$

By Lemma 2.3,

$$\frac{1}{V_o(r)} \int_{B_o(r)} w_i \le C_5 t^{-1} \int_{4r}^{\infty} s\exp\left(-\frac{s^2}{20t}\right) k_{||d\rho^{(i)}||}(s)ds$$

where $C_5 > 0$ is independent of i, t, r, provided that i is so large that $d\rho^{(i)} = d\rho = 0$ in $B_o(4r)$. Since $k_f(s) = o(s^{-1})$, then it is easy to see that

$$(6.8) \qquad \frac{1}{V_o(r)} \int_{B_o(r)} w_i \leq \epsilon(r)$$

with $\epsilon(r) \to o$ as $r \to \infty$, and the function $\epsilon(r)$ is independent of i. Let $i \to \infty$, we conclude that $d\eta = 0$. $\qquad\square$

7. Solution of Poincaré-Lelong equation

In this section we shall prove the result generalizing [13, Theorem 6.1]. With the notations of previous sections we can state the main theorem.

Theorem 7.1. *Let (M^n, g) be a complete noncompact Kähler manifold (of complex dimension n) with nonnegative Ricci curvature and nonnegative quadratic orthogonal bisectional curvature. Suppose ρ is a smooth d-closed real $(1,1)$-form on M and let $f = ||\rho||$ be the norm of ρ. Suppose that*

$$(7.1) \qquad \int_0^\infty k_f(r)dr < \infty$$

and one of following

(i) *there exist $R_s \to \infty$ and open sets Ω_s with smooth pseudo-convex boundary such that $B_o(R_s) \subset \Omega_s \subset B_o(2R_s)$;*

(ii) *M has nonnegative orthogonal bisectional curvature and $\rho + a\omega$ is nonnegative for some constant $a \geq 0$;*

(iii) *M satisfies the curvature assumption (6.5).*

Then there is a smooth function u so that $\rho = \sqrt{-1}\partial\bar{\partial}u$. Moreover, for any $0 < \epsilon < 1$, the estimate (1.3) holds.

Proof. It follows from Theorems 2.1, 4.1, 5.1, 6.1 and Proposition 2.1. $\qquad\square$

Even though it is not used in the proof, here we make some comments on relation of the conditions of *nonnegative bisectional curvature* (NB), *nonnegative orthogonal bisectional curvature* (NOB), *nonnegative quadratic orthogonal bisectional curvature* (NQOB) (5.1), and assumption (6.5) which we abbreviate it as (NCF) (nonnegativity on some 3-forms), as well as invariant representation of them. Algebraically (NB) is stronger than (NOB), which is in turn stronger than (NQOB). We introduce some notations for the convenience. First the curvature operator of a Kähler manifold can be viewed as bilinear form on $\mathfrak{gl}(n, \mathbb{C})$ (which can be identified with $\wedge^{1,1}(\mathbb{C}^n)$ via the metric) in the sense that for any $X \wedge \overline{Y}, Z \wedge \overline{W}$

$$\langle \text{Rm}(X \wedge \overline{Y}), \overline{Z \wedge \overline{W}} \rangle \doteqdot \text{Rm}(X \wedge \overline{Y}, \overline{Z \wedge \overline{W}}) = R_{X\overline{Y}W\overline{Z}}.$$

Hence for any $\Omega = (\Omega^{i\bar{j}})$, it is easy to check that $\langle \text{Rm}(\Omega), \overline{\Omega} \rangle = R_{i\bar{j}k\bar{l}}\Omega^{i\bar{j}}\overline{\Omega^{k\bar{l}}} \in \mathbb{R}$. Hence one can identify (cf. [16]) the condition (NB) as

$$(7.2) \qquad \{\text{Rm} \mid \langle \text{Rm}(\Omega), \overline{\Omega} \rangle \geq 0, \text{ for any } \Omega, \text{rank}(\Omega) = 1\}.$$

Similarly, condition (NOB) is equivalent to

$$(7.3) \qquad \{\mathrm{Rm} \mid \langle \mathrm{Rm}(\Omega), \overline{\Omega} \rangle \geq 0, \text{ for any } \Omega, \mathrm{rank}(\Omega) = 1, \Omega^2 = 0\}.$$

For the other two conditions we recall two operators from the study of the Ricci flow. The first one is $\bar{\wedge}$ operator on A, B, any two Hermitian symmetric transformations on $T'M$, defined by

$$
\begin{aligned}
(A \bar{\wedge} B)_{i\bar{j}k\bar{l}} &= A_{i\bar{j}} B_{k\bar{l}} + B_{i\bar{j}} A_{k\bar{l}} + A_{i\bar{l}} B_{k\bar{j}} + B_{i\bar{l}} A_{k\bar{j}} \\
&= 2\langle (A \wedge \bar{B} + B \wedge \bar{A})(e_i \wedge e_{\bar{j}}), \overline{e_l \wedge e_{\bar{k}}} \rangle \\
&\quad + 2\langle (A \wedge \bar{B} + B \wedge \bar{A})(e_k \wedge e_{\bar{j}}), \overline{e_l \wedge e_{\bar{i}}} \rangle.
\end{aligned}
$$

The resulting operator so-defined is also a Kähler curvature operator, in particular, satisfying the 1st-Bianchi identity. Here $(A \wedge B)(X \wedge Y) = \frac{1}{2}(A(X) \wedge B(Y) + B(X) \wedge A(Y))$ as defined in [16]. This operator is the one involved in the $\mathsf{U}(n)$-invariant irreducible decomposition of the space of the Kähler curvature operators. Now the condition (NQOB) is equivalent to

$$(7.4) \qquad \{\mathrm{Rm} \mid \langle \mathrm{Rm}, A^2 \bar{\wedge} \,\mathrm{id} - A \bar{\wedge} A \rangle \geq 0, \text{ for all Hermitian symmetric } A\}.$$

For (NCF) we need to introduce the so-called #-operator, which is defined for any two curvature operator $R_{i\bar{j}k\bar{l}}$ and $S_{i\bar{j}k\bar{l}}$, under a unitary frame, as

$$(R\#S)_{i\bar{j}k\bar{l}} = R_{i\bar{p}q\bar{l}} S_{p\bar{j}k\bar{q}} - R_{i\bar{p}k\bar{q}} S_{p\bar{j}q\bar{l}}.$$

The operator is related to the Ricci flow and was first introduced by Hamilton [16]. Direct calculation shows that (NCF) is equivalent to

$$(7.5) \qquad \{\mathrm{Rm} \mid (\mathrm{Rm} \,\#\, \mathrm{I})(\Omega, \overline{\Omega}) \geq 0, \text{ for all } \Omega \in \mathfrak{gl}(n, \mathbb{C})\}$$

where $\mathrm{I}_{i\bar{j}k\bar{l}} = g_{i\bar{j}} g_{k\bar{l}} + g_{i\bar{l}} g_{k\bar{j}}$. It has been known that (NB), (NOB) are Ricci flow invariant conditions [16]. It would be interesting to find out about (NQOB) and (NCF).

The smoothing lemma [13, Lemma 4.1] on the Busemann function and part (i) of Theorem 7.1 imply the following corollary, because of the fact that the Busemann function is comparable with the distance function, by [5, Theorem 2.3] and [15, p.400–401], as $x \to \infty$.

Corollary 7.1. *Let (M, g) be a complete noncompact Kähler manifold. Suppose M has nonnegative holomorphic bisectional curvature. Let ρ be a smooth d-closed $(1,1)$-form on M satisfying (7.1). Assume further that M is of maximal volume growth or has non-negative sectional curvature outside a compact subset. Then there exists $u(x)$ such that $\sqrt{-1}\partial\bar{\partial} u = \rho$.*

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